3.5 - two factors Analysis of variance
(two - way ANOVA)

Introduction: Let us consider that instead of one factor we have two factores that we want to study simultaneously.

Example: Y - 'percentage of smokers in a population' and Let us consider the following factors:

- Age - with 3 levels
- Income - with 5 levels

thus, we will have $3 \times 5 = 15$ combinations.

- We will denominate factor A and B with a and b levels
- We will call treatment to each combination of the Levels of factor A and B

When in the Design we consider all the possible treatments ($a \times b$ treatments, ie, combinations of Levels of factor A and B) it is called complete factorial Design - we will study this case

Instead of two-factor design the investigator could be tempted to use two separated one-way factor designs but the use of two-way ANOVA's model should be preferred because:

i) Efficiency

using two Anova's with one factor, we should (need) decided for a given factor (e.g. Age) what should be the level of the other factor (Income) considered

ii) Amount of Information

Considering two separate studies with one factor demands more observations and two separate experimentae designs

- fixing one level of Income, study the effect of Age over the % of smokers;
- fixing one level of age, study the effect of Income over the % of smokers.

iii) validity of findings

two separate analysis does not give information about possible interactions between the factors

How to organize the data to work with two-way ANOVA?

- The response variable Y is continuous.
- There are now two categorical explanatory variables (factors). Call them factor A and factor B.
- **Data for Two-way ANOVA:**
  - Y, the response variable;
  - Factor A with levels $i = 1, \ldots, a$;
  - Factor B with levels $j = 1, \ldots, b$;
  - A particular combination of levels is called a treatment or a cell. There are ab treatments;
  - Y_{ijk} is the k-th observation for treatment (i, j), $k = 1, \ldots, n$.
- We will assume equal sample size in each treatment combination, $n_{ij} = n > 1$ and $N = abn$. We have a balanced design.

We will just study the balanced design case:
$N = abn$

the data can be organized in the following table:

$$y_{i..} = \sum_{j=1}^{a} \sum_{k=1}^{n} y_{ijk}$$

Factor A	Factor B: 1	2	...	b	totals	Average
1	y_{111} y_{112} $\vdots$ y_{11n}	y_{121} y_{122} $\vdots$ y_{12n}	... ...	y_{1b1} y_{1b2} $\vdots$ y_{1bn}	$y_{1..}$	$\bar{y}_{1..} = \dfrac{y_{1..}}{b \times n}$
2	y_{211} y_{212} $\vdots$ y_{21n}	y_{221} y_{222} $\vdots$ y_{22n}	... ...	y_{2b1} y_{2b2} $\vdots$ y_{2bn}	$y_{2..}$	$\bar{y}_{2..}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		
a	y_{a11} y_{a12} $\vdots$ y_{a1n}	y_{a21} y_{a22} $\vdots$ y_{a2n}	... ...	y_{ab1} y_{ab2} $\vdots$ y_{abn}	$y_{a..}$	$\bar{y}_{a..}$
totals	$y_{.1.}$	$y_{.2.}$	...	$y_{.b.}$	$y_{...}$	
Averages	$\bar{y}_{.1.}$	$\bar{y}_{.2.}$	...	$\bar{y}_{.b.}$		$\bar{y}_{...}$

$\bar{y}_{.1.} = \dfrac{y_{.1.}}{na}$

$\bar{y}_{...} = \dfrac{y_{...}}{n \times a \times b}$

y_{ijk} = k-th observation of level i of factor A and level j factor B

$k = 1, \dots, n$

$i = 1, \dots, a$

$j = 1, \dots, b$

the observations y_{ijk} should be assigned to each treatment (ie, each cell (i,j)) randomly in order to overcome possible bias. this is called Completely random design.

Linear Model to describe this data:

1

Cell Means Model: $Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$

- μ_{ij} is the theoretical mean or expected value of all observations in cell (i,j) ;

Assumption for the errors:

$$\varepsilon_{ijk} \underset{i.i.d.}{\sim} N(0, \sigma^2) \quad , \forall\, i, j, k$$

- As a consequence of the assumptions to the error model, we have that the response variable observations are independent, and normally distributed with a mean that may depend on the levels of the factors A and B, and a variance that does not (is constant).
  - $Y_{ijk} \sim N(\mu_{ij}; \sigma^2)$ and independent.

$E(Y_{ijk}) = \mu_{ij}$

$var(Y_{ijk}) = var(\varepsilon_{ijk}) = \sigma^2$

2

Alternative:
Factor Effects Model: $Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$

- μ is the overall (grand) mean;
- τ_i is the main effect of Factor A;
- β_j is the main effect of Factor B;
- $(\tau\beta)_{ij}$ is the interaction effect between A and B. Note that $(\tau\beta)_{ij}$ is the name of a parameter and does not refer to the product of τ and β.
- A model without the interaction term, i.e., $\mu_{ij} = \mu + \tau_i + \beta_j$ is called an additive model.

Parameters Definition:

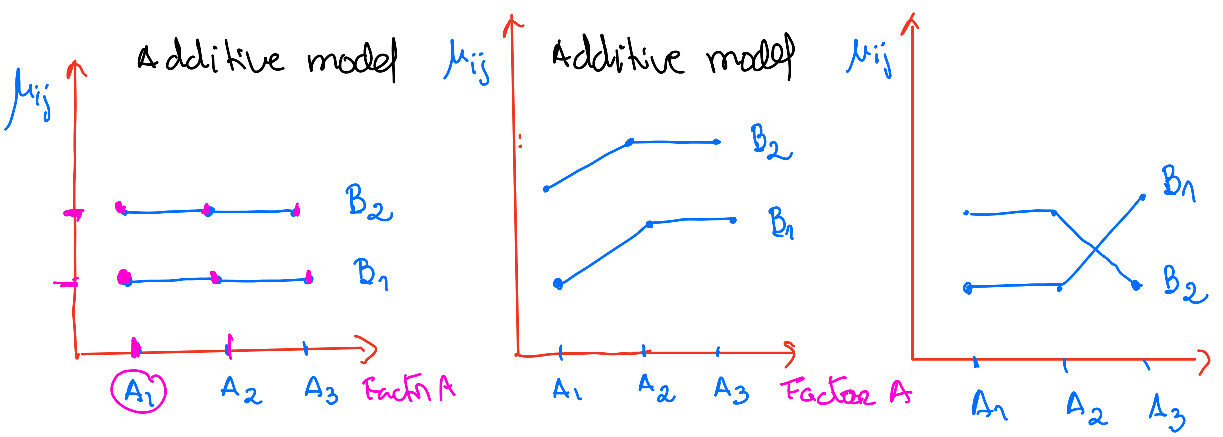
- The overall mean: $\mu = \frac{\sum_i^a \sum_j^b \mu_{ij}}{ab}$.
- The mean for the i-th level of A is $\mu_{i.} = \frac{\sum_j^b \mu_{ij}}{b}$.
- The mean for the j-th level of B is $\mu_{.j} = \frac{\sum_i^a \mu_{ij}}{a}$.
- So, $\mu_{i.} = \mu + \tau_i$ and $\mu_{.j} = \mu + \beta_j \Rightarrow$

$$\tau_i = \mu_{i.} - \mu \quad \text{and} \quad \beta_j = \mu_{.j} - \mu_{..}$$

- $(\tau\beta)_{ij} = \mu_{ij} - (\mu + \tau_i + \beta_j) = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..}$

Parameters Interpretation

- τ_i is an adjustment for level i of A and β_j is an adjustment for level j of B, related to the overall mean μ. They are called the principal effects.
- $(\tau\beta)_{ij}$ is an additional adjustment that takes into account both levels i and j. This is called the interaction effect. Non interaction effect $\Rightarrow$ additive model.



there is no effect of factor A and effect of factor B - no interaction

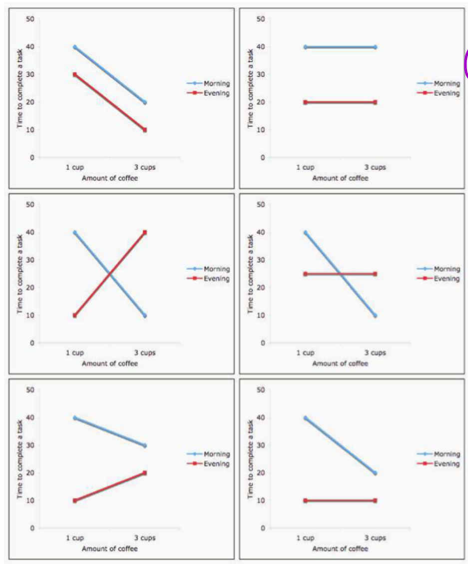
Effect of factors A and B and no interaction

effect of factors A and B + Interaction

μ_{11} μ_{21} μ_{31}

μ_{12} μ_{22} μ_{32}

Analyse existence of interaction effect:



1 2 3 4 5 6

Y = time to complete a task

Factor A - Amount of coffee

Factor B - time (Morning / evening)

1 Effect of factor A and B (no interaction) (Additive)

2 No effect of factor A but effect of factor B (no interaction) (Additive)

3 + 4 Effect of both factors and interaction

5 + 6 Effect of both factors and no interaction (Additive)

Zero-sum Constraints

- As in the one-way model, we now have too many parameters and need now several constraints:
  1. $\tau_{\cdot} = \sum_{i=1}^{a} \tau_i = 0$;
  2. $\beta_{\cdot} = \sum_{j=1}^{b} \beta_j = 0$;
  3. $(\tau\beta)_{\cdot j} = \sum_{i=1}^{a} (\tau\beta)_{ij} = 0, \quad \forall j$;
  4. $(\tau\beta)_{i\cdot} = \sum_{j=1}^{b} (\tau\beta)_{ij} = 0, \quad \forall i$.

Pontual estimates

① Cells means model: $y_{ijk} = \mu_{ij} + \varepsilon_{ijk}$

$$\hat{E}(Y_{ijk}) = \hat{\mu}_{ij} = \bar{Y}_{ij\cdot} = \sum_{k=1}^{n} \frac{Y_{ijk}}{n} = \frac{Y_{ij\cdot}}{n}$$

② Factor effect model:

$$\hat{E}(Y_{ijk}) = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + (\hat{\tau\beta})_{ij}$$

$$\hat{\mu} = \bar{Y}_{\cdots} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} Y_{ijk}}{N} = \frac{Y_{\cdots}}{N}$$

$$\hat{\tau}_i = \hat{\mu}_{i\cdot} - \hat{\mu} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots} = \frac{Y_{i\cdot\cdot}}{bn} - \frac{Y_{\cdots}}{N}$$

$$\hat{\beta}_j = \hat{\mu}_{\cdot j} - \hat{\mu} = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdots} = \frac{Y_{\cdot j\cdot}}{an} - \frac{Y_{\cdots}}{N}$$

$$(\hat{\tau\beta})_{ij} = \hat{\mu}_{ij} - \hat{\tau}_i - \hat{\beta}_j - \hat{\mu}$$

$$= \hat{\mu}_{ij} - \hat{\mu}_{i\cdot} + \hat{\mu} - \hat{\mu}_{\cdot j} + \cancel{\hat{\mu}} - \cancel{\hat{\mu}}$$

$$= \hat{\mu}_{ij} - \hat{\mu}_{i\cdot} - \hat{\mu}_{\cdot j} + \hat{\mu}$$

$$= \bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdots}$$

$$\sum_i (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$$

Variance Estimator

$$\widehat{Var}(y_{ijk}) = \hat{\sigma}^2 = \frac{SST}{N-1} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2}{N-1} = $$

$SST = MST$

$$= \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - N\bar{y}_{...}^2}{N-1}$$

Residuals

$$e_{ijk} = (y_{ijk} - \bar{y}_{ij.})$$

where $\bar{y}_{ij.}$ is the cell average

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

Hypothesis that we will test:

1. No main effect of factor A $(\mu_{i\cdot} = \mu + \tau_i)$

$H_0: \mu_{1\cdot} = \mu_{2\cdot} = \cdots = \mu_{a\cdot}$

$\iff \tau_1 = \tau_2 = \cdots = \tau_a = 0$

$H_1: \exists (i,j): \mu_{i\cdot} \neq \mu_{j\cdot}$

$\iff$ At least one $\tau_i \neq 0$

2. No main effect of factor B $(\mu_{\cdot j} = \mu + \beta_j)$

$H_0: \mu_{\cdot 1} = \mu_{\cdot 2} = \cdots = \mu_{\cdot b}$

$\iff \beta_1 = \beta_2 = \cdots = \beta_b = 0$

$H_1: \exists (i,j): \mu_{\cdot i} \neq \mu_{\cdot j}$

$\iff$ At least one $\beta_j \neq 0$

3. No interaction

(Additive model)

$H_0: (\tau\beta)_{11} = (\tau\beta)_{12} = \cdots = (\tau\beta)_{ab} = 0$

$H_1:$ At least one $(\tau\beta)_{ij} \neq 0$

$SST = SSE + SSTR$

the test statistics are based on the decomposition of the total variability in components

$$(y_{ijk}-\bar{y}_{\cdots}) = (y_{ijk}-\bar{y}_{ij\cdot}) + (\bar{y}_{ij\cdot}-\bar{y}_{\cdots})$$

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{\cdots})^2}_{SST} = \underbrace{\sum_{i}\sum_{j}\sum_{k}(y_{ijk}-\bar{y}_{ij\cdot})^2}_{SSE} + \underbrace{\sum_{i}\sum_{j}\sum_{k}(\bar{y}_{ij\cdot}-\bar{y}_{\cdots})^2}_{SSTR}$$

and SSTR can be decomposed as:

$$SSTR = SSA + SSB + SS\,AB$$

$$SSA = \sum_i\sum_j\sum_k \hat{\tau}_i^2 = \sum_i\sum_j\sum_k \left(\frac{y_{i\cdot\cdot}}{bn} - \frac{y_{\cdots}}{N}\right)^2$$

$$= \sum_i\sum_j\sum_k \left(\frac{y_{i\cdot\cdot}}{bn}\right)^2 - N\left(\frac{y_{\cdots}}{N}\right)^2 =$$

$$= \sum_i\sum_j\sum_k \frac{y_{i\cdot\cdot}^2}{b^2n^2} - \frac{y_{\cdots}^2}{N} = bn\sum_i \frac{y_{i\cdot\cdot}^2}{b^2n^2} - \frac{y_{\cdots}^2}{N} =$$

$$= \sum_i \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdots}^2}{N}$$

$$SSB = \sum_i \sum_j \sum_k \hat{\beta}_j^2 = \sum_i \sum_j \sum_k \left(\frac{y_{.j.}}{ab} - \frac{y_{...}}{N} \right)^2 =$$

$$= \cdots = \sum_j \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{N}$$

$$SSAB = \sum_i \sum_j \sum_k (\widehat{\tau\beta})_{ij}^2$$

We can use the fact that:

$$(\widehat{\tau\beta})_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...} =$$

$$= (\bar{y}_{ij.} - \bar{y}_{...}) + (\bar{y}_{...} - \bar{y}_{i..}) - (\bar{y}_{.j} - \bar{y}_{...})$$

$$= (\bar{y}_{ij.} - \bar{y}_{...}) - \underbrace{(\bar{y}_{i..} - \bar{y}_{...})} - \underbrace{(\bar{y}_{.j.} - \bar{y}_{...})}$$

$$= (\bar{y}_{ij} - \bar{y}_{...}) - \text{term related with Factor A} - \text{term related with Factor B}$$

$$SSAB = \sum_i \sum_j \sum_k (\widehat{\tau\beta})_{ij}^2 = \sum_i \sum_j \frac{y_{ij}^2}{n} - \frac{y_{...}^2}{N} =$$

$$- SSA - SSB$$

SS for ANOVA Table

$$
\begin{aligned}
SSA &= \sum_i^a \sum_j^b \sum_k^n \hat{\tau}_i{}^2 = \sum_i^a \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdot\cdot\cdot}^2}{abn}; \\
SSB &= \sum_i^a \sum_j^b \sum_k^n \hat{\beta}_j{}^2 = \sum_j^b \frac{y_{\cdot j\cdot}^2}{an} - \frac{y_{\cdot\cdot\cdot}^2}{abn}; \\
SSAB &= \sum_i^a \sum_j^b \sum_k^n (\widehat{\tau\beta})_{ij}^2 = \sum_i^a \sum_j^b \frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdot\cdot\cdot}^2}{abn} - SSA - SSB; \\
SSE &= \sum_i^a \sum_j^b \sum_k^n \left(Y_{ijk} - \bar{Y}_{ij\cdot}\right)^2; \\
SST &= \sum_i^a \sum_j^b \sum_k^n \left(Y_{ijk} - \bar{Y}_{\cdot\cdot\cdot}\right)^2 = \sum_i^a \sum_j^b \sum_k^n y_{ijk}^2 - \frac{y_{\cdot\cdot\cdot}^2}{abn} \\
&= SSA + SSB + SSAB + SSE.
\end{aligned}
$$

ANOVA Table

Source of variation	SS	df	MS
A Treatments	$SSA = \sum_i^a \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdot\cdot\cdot}^2}{abn}$	$a-1$	$MSA = \frac{SSA}{a-1}$
B Treatments	$SSB = \sum_j^b \frac{y_{\cdot j\cdot}^2}{an} - \frac{y_{\cdot\cdot\cdot}^2}{abn}$	$b-1$	$MSB = \frac{SSB}{b-1}$
Interaction	$SSAB = \sum_i^a \sum_j^b \frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdot\cdot\cdot}^2}{abn} - SSA - SSB$	$(a-1)(b-1)$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$
Error	$SSE = \sum_i^a \sum_j^b \sum_k^n (y_{ijk} - \bar{y}_{ij\cdot})^2$	$ab(n-1)$	$MSE = \frac{SSE}{ab(n-1)}$ $\hat{\sigma}^2$
Total	$SST = \sum_i^a \sum_j^b \sum_k^n y_{ijk}^2 - \frac{y_{\cdot\cdot\cdot}^2}{abn}$	$abn-1$	

Test Hypotheses for two-way ANOVA

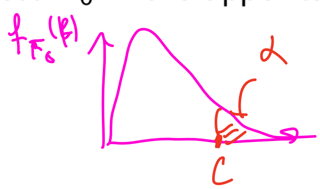
- Test for factor A effect:

$$H_0 : \mu_{1.} = \mu_{2.} = \cdots = \mu_{a.} \;\; vs \;\; H_1 : \mu_{i.} \neq \mu_{j.} \; \exists_{(i,j)}$$

$$\Rightarrow H_0 : \tau_i = 0 \,, \forall i \;\; vs \;\; H_1 : \tau_i \neq 0 \,, \exists_i$$

Under H_0 we have that $F_0 = \dfrac{MSA}{MSE} \overset{H_0}{\sim} F_{(a-1, ab(n-1))}$.

- We reject H_0 in the upper-tailed of the $F_{(a-1,ab(n-1))}$ distribution.



- Test for factor B effect:

$$H_0 : \mu_{.1} = \mu_{.2} = \cdots = \mu_{.b} \;\; vs \;\; H_1 : \mu_{.i} \neq \mu_{.j} \; \exists_{(i,j)}$$

$$\Rightarrow H_0 : \beta_j = 0 \,, \forall j \;\; vs \;\; H_1 : \beta_j \neq 0 \,, \exists_j$$

Under H_0 we have that $F_0 = \dfrac{MSB}{MSE} \overset{H_0}{\sim} F_{(b-1, ab(n-1))}$.

- We reject H_0 in the upper-tailed of the $F_{(b-1,ab(n-1))}$ distribution.

- Test for interaction Effect:

$$H_0 : (\tau\beta)_{ij} = 0 \ , \forall (i,j) \ \ vs \ \ H_1 : (\tau\beta)_{ij} \neq 0 \ , \exists_{(i,j)}$$

Under H_0 we have that $F_0 = \dfrac{MSAB}{MSE} \overset{H_0}{\sim} F_{((a-1)(b-1),ab(n-1))}$.

- We reject H_0 in the upper-tailed of the $F_{((a-1)(b-1),ab(n-1))}$ distribution.

Example:

A pharmaceutical company is testing a new compound for hash fever relief. In a preliminary trial with 36 volunteers the concentrations of the two main ingredients were varied in three levels each. The volunteers were randomly distributed by the nine combinations and the number of hours of relief were measured. The results are shown in the next tables.

y	Factor B (ingredient 2)		
Factor A (ingredient 1)	$j=1$ low	$j=2$ medium	$j=3$ high
$i=1$ low	2.4 2.7 2.3 2.5	4.6 4.2 4.9 4.7	4.8 4.5 4.4 4.6
$i=2$ medium	5.8 5.2 5.5 5.3	8.9 9.1 8.7 9.0	9.1 9.3 8.7 9.4
$i=3$ high	6.1 5.7 5.9 6.2	9.9 10.5 10.6 10.1	13.5 13.0 13.3 13.2

The means ($\bar{y}_{ij\cdot}$) and standard deviations (s_{ij}) in each cell are:

$\bar{y}_{ij\cdot};s_{ij}$	Factor B		
Factor A	$j=1$	$j=2$	$j=3$
$i=1$	2.475 ; .1708	4.600 ; .2944	4.575 ; .1708
$i=2$	5.450 ; .2646	8.925 ; .1708	9.125 ; .3096
$i=3$	5.975 ; .2217	10.275 ; .3304	13.250 ; .2082

1. Describe the ANOVA model that is adequate to analyze this experiment. Do not forget to mention the assumptions of the model. (

2. Use a graphical representation of the means to assess (qualitatively) the effects of the two factors and of their interaction.

3. Use the following information: $SST = 374.73$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{4}\left(y_{ijl} - \bar{y}_{ij\cdot}\right)^2 = 1.625 \qquad \sum_{i=1}^{3}\left(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}\right)^2 = 18.335 \qquad \sum_{j=1}^{3}\left(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}\right)^2 = 10.305$$

to compute the ANOVA table and perform the associated F tests. Are your conclusions in agreement with the graphical analysis from the previous question?

Example: Sales of Bread

An experimental study was made to check the effect of height of the shelf display (Factor A: bottom ($i=1$), middle ($i=2$), top ($i=3$)) and the width of the shelf display (Factor B: regular ($j=1$), wide ($j=2$)) on the sales of the bakery's bread during a experimental period.

12 supermarkets similar in terms of sales volume and clientele were utilized in the study and the 6 treatments (3x2) were assigned at random to each of 2 stores. Sales of bread were recorded with the results:

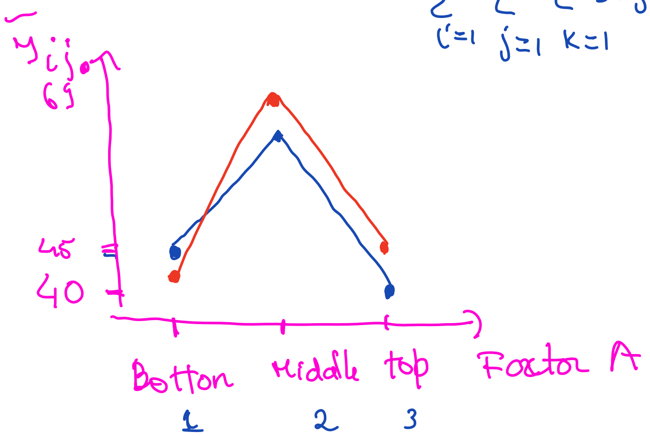
Y_{ijk} = sales of bread of the k-store with bread displayed in a shelf on height i and width j

$k = 1,2$; $i = 1,2,3$; $j = 1,2$

Factor A (display height)	Factor B (display width) regular $j=1$	wide $j=2$	$y_{i..}$	$\bar{y}_{i..}$
Bottom $i=1$	47 y_{111} 43 y_{112} $y_{11.} = 90$	46 40 $y_{12.} = 86$	176	44
$i=2$ Middle	62 68 $y_{21.} = 130$	67 71 $y_{22.} = 138$	268	67
$i=3$ top	41 39 $y_{31.} = 80$	42 46 $y_{32.} = 88$	168	42
$y_{.j.}$	300	312	$612 = \sum_i \sum_j \sum_k y_{ijk}$ = $y_{...}$	
$\bar{y}_{.j.}$	50	52		$51 = \bar{y}_{...}$

Factor A (display height)	Factor B (display width) regular	wide	$y_{i..}$	$\bar{y}_{i..}$
Bottom	47 43 $y_{11.} = 90$	46 40 $y_{12.} = 86$	176	44
Middle	62 68 $y_{21.} = 130$	67 71 $y_{22.} = 138$	268	67
top	41 39 $y_{31.} = 80$	42 46 $y_{32.} = 88$	168	42
$y_{.j.}$	300	312	$612 = y_{...}$	
$\bar{y}_{.j.}$	50	52		$51 = \bar{y}_{...}$

$$\sum_{i=1}^{3} \sum_{j=1}^{2} \sum_{k=1}^{2} y_{ijk}^{2} = 32854$$



- regular $(j=1)$
- wide $(j=2)$

$\bar{y}_{11.} = 45 = \frac{90}{2}$

$\bar{y}_{21.} = 130/2 = 65$

$\bar{y}_{31.} = 80/2 = 40$

$\bar{y}_{12.} = 86/2 = 43$

$\bar{y}_{22.} = 138/2 = 69$

$\bar{y}_{32.} = 88/2 = 44$

$\exists$ interaction + main factor A + main factor B

Anova table:

$N = 12$

$$SST = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{N} = 32854 - \frac{(612)^2}{12}$$

$$= 1642$$

$$SSA = \sum_i \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{N} = \frac{176^2 + 268^2 + 168^2}{4} - \frac{(612)^2}{12}$$

$$= 1544$$

$$SSB = \sum_j \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{N} = \frac{300^2 + 312^2}{6} - \frac{(612)^2}{12}$$

$$= 12$$

$$SSAB = \sum_i \sum_j \frac{y_{ij.}}{n} - \frac{y_{...}^2}{abn} - SSA - SSB$$

$$= \frac{90^2 + 86^2 + 130^2 + 138^2 + 80^2 + 88^2}{2} - \frac{612^2}{12} - 1544 - 12 = 24$$

ANOVA table:

Source of variations	SS	df	MS	$f_0 = \frac{MS}{MSE}$
treat A	1544	$a-1 = 2$	772	74.710
treat B	12	$b-1 = 1$	12	1.161
Interactions AB	24	$(a-1)(b-1) = 2$	12	1.61
Error	62	$ab(n-1) = 6$	10.383	
total	1642	$abn-1 = 11$		

$$SSE = SST - SSA - SSB - SSAB$$

testing for interactions and main effects with $\alpha = 0.05$

$$\text{Model: } Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

$i = 1, \dots, 3$ $\quad$ $\mu \to$ over all mean of Y

$j = 1, 2$ $\quad$ $\tau_i \to$ main effect of factor A

$k = 1, 2$ $\quad$ $\beta_j \to$ " " " " B

$(\tau\beta)_{ij} \to$ interaction effect

ε_{ijk} - error (Random)

Assumption: $\varepsilon_{ijk} \underset{i.i.d.}{\sim} N(0, \sigma^2)$

restriction: $\sum_i \tau_i = 0$; $\sum_j \beta_j = 0$, $\sum_i (\tau\beta)_{ij} = 0$

$$\sum_j (\tau\beta)_{ij} = 0$$

Hypotheses:

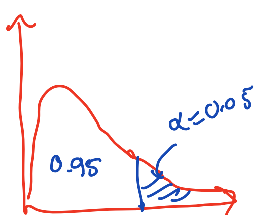
1) Interactions (AB)

$H_0: (\tau\beta)_{ij} = 0 \ \forall i,j$

$H_1: \exists_{i,j} : (\tau\beta)_{ij} \neq 0$

test. Statistic:

$$F_{0_{AB}} = \frac{MSAB}{MSE} \sim F_{(2,6)}$$



$c = F^{-1}_{F(2,6)}(0.95) = 5.14$

$CR =]5.14; +\infty[$

$f_{0_{AB}} = 74.710 \in C.R$

$\Rightarrow$ rej H_0, for $\alpha = 0.05$, the interaction exist.

2) Main effect A

$H_0: \tau_i = 0, \forall i$

$H_1: \exists_i : \tau_i \neq 0$

test statistic:

$$F_{0_A} = \frac{MSA}{MSE} \sim F_{(2,6)}$$

$C.R =]5.14; +\infty[$

$f_{0_A} = 1.161 \notin CR$

$\bar{n}$ rej H_0, not main effect factor A

3) Main effect fact.B

$H_0: \beta_j = 0, \forall j$

$H_1: \exists j : \beta_j \neq 0$

test statistic:

$$F_{0_B} = \frac{MSB}{MSE} \sim F_{(1,6)}$$

$CR =]5.99; +\infty[$

$f_{0_B} = 1.161 \notin CR$

$\bar{n}$ rej H_0, not main effect factor B